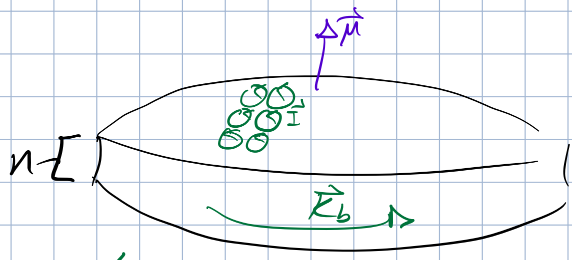
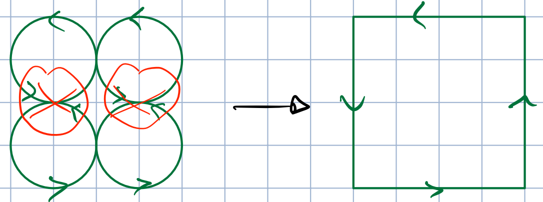
Magnetized Material

① Uniformly Magnetized

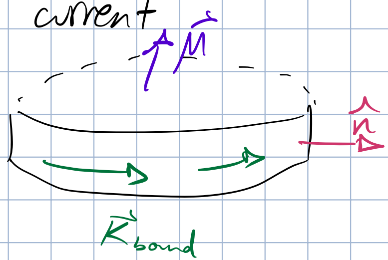


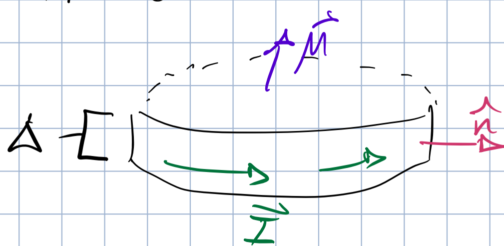
from top 

internal currents cancel out.

edge charges move only in tiny loop → net effect is macroscopic

Bound current = "Free" current





$\vec{K}_b = \vec{\mu} \times \hat{n}$

$K = \frac{I}{\Delta}$

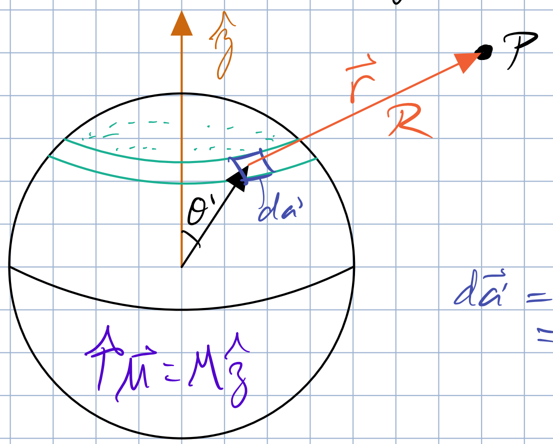
② Nonuniform Magnetized

internal currents don't cancel now :(

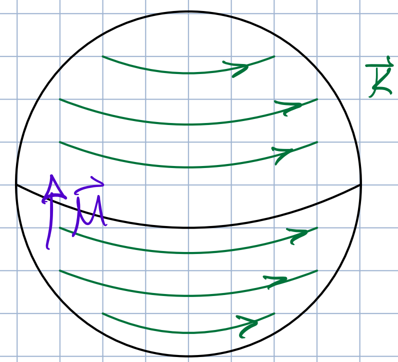
$$\vec{J}_{bound} = \vec{\nabla} \times \vec{\mu}$$

$$\vec{\nabla} \cdot \vec{J} = 0$$

Example: Uniformly Magnetized Sphere



$d\vec{a}' = da' \cdot \hat{n}'$
$= da'\,\hat{r}'$



$\vec{K}_b = \vec{M} \times \hat{n}$
$= M \sin\theta \cdot \vec{\phi}$

$$\vec{J}_b = \nabla \times \vec{M} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{n} = \vec{M} \times \hat{r}' = M \sin\theta' d\hat{\phi}'$$

$$\rightarrow A(\vec{r}) = \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b}{R}\, da'$$

$$= \frac{\mu_0}{4\pi} \oint \frac{\vec{M} \times \hat{n}}{R}\, da'$$

$$= \frac{\mu_0}{4\pi} \oint \frac{M \sin\theta' \hat{\phi}'}{R}\, da'$$

$$= \frac{\mu_0 M \hat{\phi}'}{4\pi} \oint \frac{\sin\theta'\, da'}{R}$$

Say it spins $\vec{\omega}$ counter clockwise

$$\vec{K} = \sigma \cdot \vec{v} = \sigma \cdot v \cdot \hat{\phi}'$$

period: $T = \frac{2\pi a \sin\theta'}{v}$

$\omega = 2\pi f = \frac{v}{a \sin\theta'}$ $\rightarrow$ $v = \omega a \sin\theta'$

$$\vec{K} = \sigma \cdot a \sin\theta' \cdot \omega \cdot \hat{\phi}'$$

$$\vec{K} = \sigma \cdot a \cdot \vec{\omega} \times \hat{r}$$

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b}{R}\, da' = \frac{\mu_0 a \sigma}{4\pi} \oint \frac{\vec{\omega} \times \hat{r}'}{R}\, da'$$

Like in ex. 5.11   pick $\hat{r} = \hat{z}$ & $\vec{\omega}$ on x-z plane

$$\mathcal{R} = \sqrt{r^2 + a^2 - 2ar\cos\theta'}$$

$\vec{\omega}$ & $\hat{z}$

$$\vec{\omega} \times \hat{r}' = \hat{x}(-\cos\alpha \sin\theta' \cancel{\sin\phi'})\omega + \hat{y}(\cos\alpha \sin\theta' \cancel{\cos\phi'} - \sin\alpha\cos\theta')\omega + \hat{z}(\sin\alpha \sin\theta' \cancel{\sin\phi'})\omega$$

$$\int_0^{2\pi} \cos\phi' \, d\phi' = \int_0^{2\pi} \sin\phi' \, d\phi' = 0 \quad \star$$

$$\vec{\omega} \times \hat{r}' = -\omega \sin\alpha \cos\theta' \; \hat{y}$$

$$A(\vec{r}) = \frac{\mu_0 \sigma a}{4\pi} \int \frac{-\omega \sin\alpha \cos\theta'}{\mathcal{R}} \, da' \; \hat{y}$$

$$= -\frac{\mu_0 \sigma a \omega \sin\alpha}{4\pi} (\hat{y}) \int \frac{\cos\theta'}{\sqrt{r^2 + a^2 - 2ar\cos\theta'}} \, a^2 \sin\theta' \, d\theta' \, d\phi'$$

$$= \frac{-\mu_0 \sigma a^3 \omega \sin\alpha}{2} \; \hat{y} \; \underbrace{\int_0^{\pi} \frac{\cos\theta' \sin\theta' \, d\theta'}{\sqrt{r^2 + a^2 - 2ar\cos\theta'}}}_{\theta \doteq}$$

$$\longrightarrow = -\frac{1}{3a^2r^2} \left[(r^2 + a^2 + ar)\,|r-a| - (r^2 + a^2 - ar)\,|r+a| \right] = \star$$

inside) $r \leq a$

$$\star = -\frac{1}{3a^2r^2} \left[(r^2 + a^2 + ar)(a-r) - (r^2 + a^2 - ar)(r+a) \right]$$

$$= -\frac{1}{3a^2r^2} \left[-2(r^2 + a^2)r + 2ar^2 \right]$$

$$= \frac{2r}{3a^2}$$

outside) $r \geq a$

$$\star = -\frac{1}{3a^2r^2} \left[(r^2 + a^2 + ar)(r-a) - (r^2 + a^2 - ar)(r+a) \right]$$

$$= -\frac{1}{3a^2r^2} \left(2ar^2 - 2a(r^2 + a^2) \right)$$

$$= \frac{2a}{3r^2}$$

$$A(\vec{r}) = -\hat{y} \cdot \mu_0 \sigma a^3 \omega \sin\alpha \cdot \begin{cases} \frac{r}{3a^2} & \text{if } r \leq a \\ \frac{a}{3r^2} & \text{if } r \geq a \end{cases}$$

$$\vec{\omega} \times \vec{r} = -\omega r \sin\alpha$$

$$A(\vec{r}) = \begin{cases} \frac{\mu_0 \sigma a}{3} (\vec{\omega} \times \vec{r}) & \text{if } r \leq a \\ \frac{\mu_0 \sigma a^4}{3r^3} (\vec{\omega} \times \vec{r}) & \text{if } r \geq a \end{cases}$$

$$\vec{K} = \sigma \cdot a \cdot \vec{\omega} \times \hat{r}$$

$$\vec{K}_b = \vec{M} \times \hat{n}$$

$$\vec{M} \times \hat{n} = \sigma \cdot a \, \vec{\omega} \times \hat{r} = \vec{M} \times \hat{r}$$

$$A(\vec{r}) = \begin{cases} \frac{\mu_0}{3} \cdot \vec{M} \times \vec{r} & \text{if } r \leq a \\ \frac{\mu_0}{3} \cdot \frac{a^3}{r^3} \cdot \vec{M} \times \vec{r} & \text{if } r \geq a \end{cases}$$

$$= \begin{cases} \frac{\mu_0}{3} M r \sin\theta \cdot \hat{\phi} & \text{if } r \leq a \\ \frac{\mu_0}{3} \frac{a^3}{r^2} \cdot M \sin\theta \, \hat{\phi} & \text{if } r \geq a \end{cases}$$

Ok, what about $\vec{B}$?

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$A_r = 0 \qquad A_\theta = 0$$

$$\vec{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) \cdot \hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r \cdot A_\phi)\,\hat{\theta}$$

inside sphere $r \leq a$:

$$\vec{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\frac{\mu_0}{3} M r \sin\theta\right)\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \left(\frac{\mu_0}{3} M r \sin\theta\right)\right)\hat{\theta}$$

$$= \frac{\mu_0 M}{3}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta)\,\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r^2\sin\theta)\,\hat{\theta}\right)$$

$$= \frac{\mu_0 M}{3} \cdot \left(\frac{2r\sin\theta}{r\sin\theta}\cos\theta\,\hat{r} - \frac{2r}{r}\sin\theta\,\hat{\theta}\right)$$

$$= \frac{2\mu_0 M}{3}\underbrace{\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)}_{\text{math}}$$

$$= \frac{2\mu_0 M}{3}\hat{z}$$

$$\vec{B} = \frac{2\mu_0}{3}\vec{M}$$ constant & parallel to $\vec{M}$!

Outside sphere $r \geq a$:

$$\vec{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\frac{\mu_0}{3}\frac{a^3}{r^2}M\sin\theta\right)\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\left(\frac{\mu_0}{3}\frac{a^3}{r^2}M\sin\theta\right)\right)\hat{\theta}$$

$$= \frac{\mu_0 M a^3}{3}\left(\frac{1}{r\sin\theta}\cdot\frac{1}{r^2}\frac{\partial}{\partial\theta}(\sin^2\theta)\,\hat{r} - \frac{1}{r}\cdot\sin\theta\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\theta}\right)$$

$$= \frac{\mu_0 M a^3}{3}\left(\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right)$$

$$= \frac{\mu_0 M a^3}{3}\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

field produced by perfect magnetic dipole located @ center of sphere w/ $\vec{m} = \frac{4}{3}\pi a^3 \vec{M}$

Electricity & Magnetism

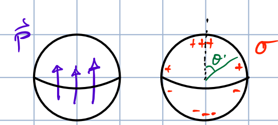
E electric dipole moments: $d\,\vec{\uparrow}$ $\quad \vec{p} = q\vec{d} \quad \vec{N} = \vec{p}\times\vec{E}$

M magnetic dipole moments: I, $\vec{A}$ $\quad \vec{m} = I\vec{A} \quad \vec{N} = \vec{m}\times\vec{B}$

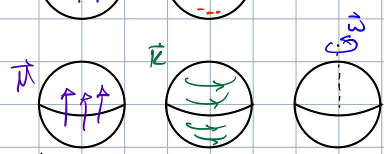
E Bound charges: $\rho_b = -\vec{\nabla}\cdot\vec{P} \quad \sigma_b = \vec{P}\cdot\hat{n}$

M Bound currents: $\vec{J}_b = \vec{\nabla}\times\vec{M} \quad \vec{K}_b = \vec{M}\times\hat{n}$

E $\vec{E}$ fields produced by uniformly polarized object w/ $r=a$
= $\vec{E}$ fields produced by spherical shell w/ $\sigma = P\cos\theta'$

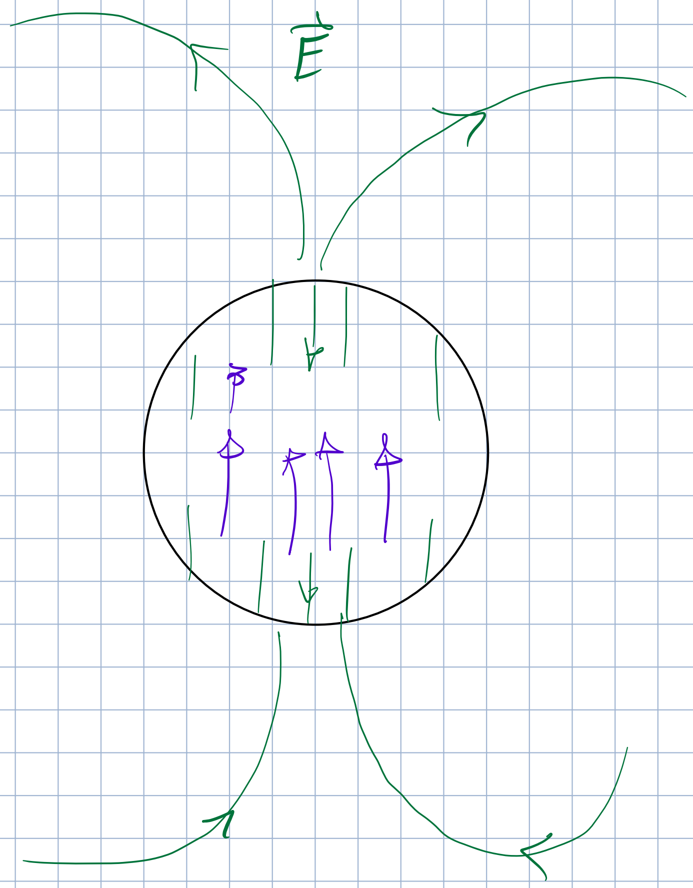


M $\vec{B}$ fields produced by uniformly magnetized sphere w/ $r=a$
= $\vec{B}$ fields produced by a surface current w/ $\vec{K} = M\sin\theta'\,\hat{\phi}$
= $\vec{B}$ fields produced by a rotating sphere w/ uniform σ & $\vec{\omega} = \frac{1}{\sigma a}\vec{M}$

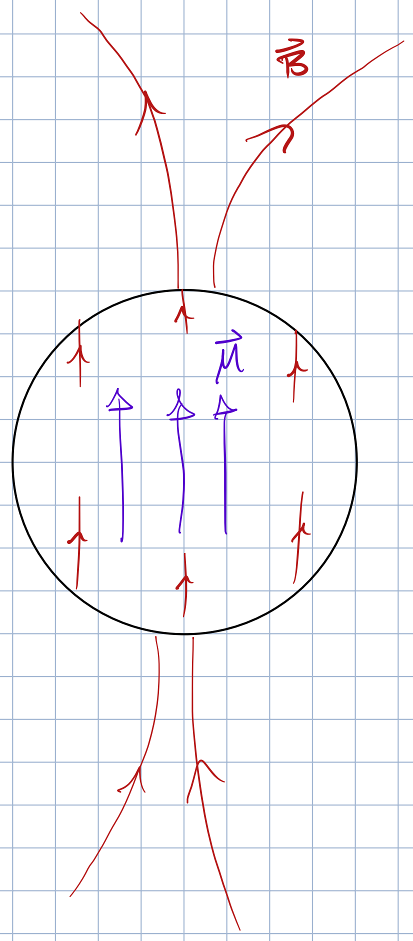


E $\vec{E}_{inside} = -\frac{1}{3\epsilon_0}\vec{P}$ constant, opposite to polarization
$\vec{E}_{outside} = \vec{E}$ produced by $\vec{E}$ dipole w/ $\vec{p} = \frac{4}{3}\pi a^3\vec{P}$

M $\vec{B}_{inside} = \frac{2\mu_0}{3}\vec{M}$ constant, parallel to magnetization
$\vec{B}_{outside} = \vec{B}$ produced by $\vec{B}$ dipole w/ $\vec{m} = \frac{4}{3}\pi a^3\vec{M}$



$\vec{E}$
$\vec{P}$



$\vec{B}$
$\vec{M}$