KPTC 106

Integrals

1-D integral

line integral

surface integral

volume integral

Fundamental Thm of

Gradients:

(local) (global)

$$\int_{\vec{a}\ \text{path}}^{\vec{b}} \vec{\nabla} T \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

independent of path! just endpoints

$\oint \vec{\nabla} \cdot T \, d\vec{l} = 0$ for closed loop

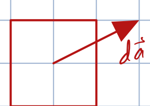
$\vec{\nabla} T$ is special vector

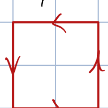
example: $T = -V$ (potential) $\vec{\nabla} T = -\vec{\nabla} V = \vec{E}$ (E field)

Curls:

local global

$$\int_{\text{surface}} (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_{\text{boundary}} \vec{A} \cdot d\vec{l}$$





relates vector field on boundary line to behavior of field on surface

Stoke's Thm

Divergence:

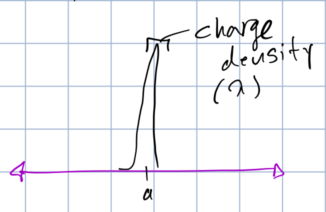
local global

$$\int_{\text{Volume}} (\vec{\nabla} \cdot \vec{A}) \, d\tau = \oint_{\text{Surface}} \vec{A} \cdot d\vec{a}$$

integral of a divergence of a vector fn over Volume = integral of same vector fn over surface bounding the volume

Gauss's Law

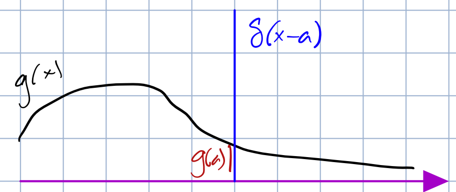
Dirac Delta F$^{\underline{n}}$



$$\int_{-\infty}^{\infty} \lambda(x)\,dx = \begin{cases} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1$$

$$\delta(x-a) = \begin{cases} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{cases}$$



$$\int_{-\infty}^{\infty} g(x)\,\delta(x-a)\,dx = g(a)$$

in 3D:

$$\delta^3(\vec{r}) \equiv \delta(x)\cdot\delta(y)\cdot\delta(z)$$

$$\rho(\vec{r}) = \rho(x,y,z) = q\,\delta(x-a_1)\,\delta(y-a_2)\,\delta(z-a_3) = q\,\delta(\vec{r}-\vec{a})$$

step f=

$$\theta(x-a) = \begin{matrix} 1 & x > a \\ 0 & x < a \end{matrix}$$

$$\frac{d\theta(x-a)}{dx} = \delta(x-a)$$

$$\frac{1}{4\pi}\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \delta^3(\vec{r}) \Rightarrow \vec{\nabla}\left(\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\right) = \frac{q}{\epsilon_0}\delta^3(\vec{r})$$

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{\epsilon_0}\delta^3(\vec{r}-\vec{a})$$ point charge

Divergence Th$^{\underline{m}}$

on sphere, $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$

$$\int_V \vec{\nabla}\cdot\vec{E}\,d\tau = \int_V \frac{q}{\underline{4\pi}\,\epsilon_0}\,\underline{\vec{\nabla}\,\frac{\hat{r}}{r^2}}\,dr$$

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q}{4\pi\epsilon_0}\oint\frac{\hat{r}}{r^2}\,dr$$

$$d\vec{a} = da\,\hat{r} = r^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{q}{4\pi\epsilon}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin\theta\,d\theta\,d\phi\,\frac{r^2}{r^2}$$

$$= \frac{q}{\epsilon_0} = \int_{\text{allspace}}\frac{q}{\epsilon_0}\,\underline{\delta^3(\vec{r})}\,d\tau$$

$$\frac{1}{4\pi}\vec{\nabla}\frac{\hat{r}}{r^2} = \delta^3(\vec{r})$$