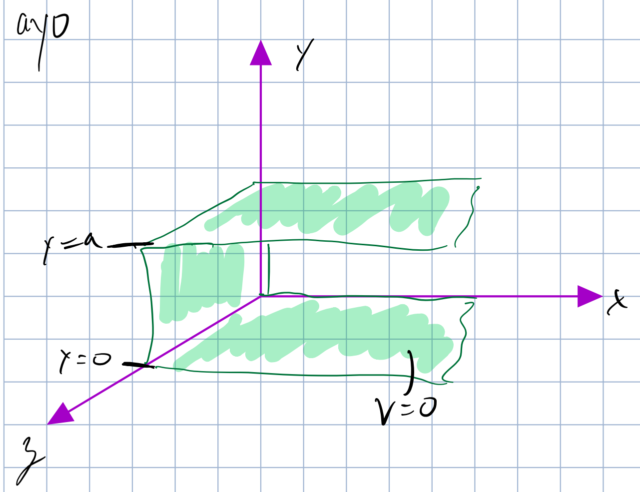
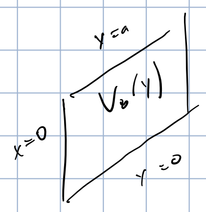


$V=0$ @ $y=0$ & a



Boundary Conditions

$V=0$ @ $y=0$
$V=0$ @ $y=a$
$V(0,y) = V_0(y)$
$V(\infty,y) = 0$ as $x \to \infty$

$$\nabla^2 V(x,y)=0 \to \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V \equiv f(x)\, g(y) \qquad \frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2}$$

(a) $\to$ α^2 $\quad -\alpha^2$
(b) $\to$ $-\alpha^2$ $\quad \alpha^2$

choose (b)

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\alpha^2 \qquad \frac{1}{g}\frac{d^2 g}{dy^2} = \alpha^2$$

$$f(x) = a_1 e^{(i\alpha)x} + a_2 e^{-(i\alpha)x}$$
$$g(y) = b_1 e^{\alpha y} + b_2 e^{-\alpha y}$$

$$V(x,y) = \sum_{\alpha>0} \underbrace{\left(a_1^\alpha e^{(i\alpha)x} + a_2^\alpha e^{-i\alpha x}\right)}_{\text{condition (4)} \to \neq 0,\ \text{so no}}\left(b_1^\alpha e^{\alpha y} + b_2^\alpha e^{-\alpha y}\right)$$

choose a

$$\frac{1}{f}\frac{d^2 f}{dx^2} = \alpha^2 \qquad \frac{1}{g}\frac{d^2 g}{dy^2} = -\alpha^2$$

$$f(x) = a_1 e^{(\alpha)x} + a_2 e^{-(\alpha)x}$$
$$g(y) = b_1 e^{i\alpha y} + b_2 e^{-i\alpha y}$$

$$V(x,y) = \sum_{\alpha>0} \left(a_1^\alpha e^{(\alpha)x} + a_2^\alpha e^{-\alpha x}\right)\left(b_1^\alpha e^{i\alpha y} + b_2^\alpha e^{-i\alpha y}\right)$$

condition 4 : $V(\infty, y) = 0 \to a_1^\alpha = 0$

$$V(x,y) = \left(a_2^\alpha e^{-\alpha x}\right)\left(b_1^\alpha e^{i\alpha y} + b_2^\alpha e^{-i\alpha y}\right)$$

condition 1 : $V(x,0) = 0 \to b_2^\alpha = -b_1^\alpha$

$$C_\alpha \equiv a_2^\alpha \cdot b_1^\alpha$$

$$V(x,y) = \sum C_\alpha e^{-\alpha x} \left(\underline{e^{i\alpha y} - e^{-i\alpha y}} \right)$$

$$= \sum C_\alpha e^{-\alpha x} \, \underline{2i \sin(\alpha y)}$$

condition 2 $\quad V(x,a) = 0$

$$\sum C_\alpha e^{-\alpha x} \, 2i \sin(\alpha a) = 0$$

$$\sin(\alpha a) = 0$$

$$\alpha = \frac{n\pi}{a} \qquad n = 0, 1, \ldots$$

$$C_{n\pi/a} \cdot 2i \equiv A_n$$

$$V(x,y) = \sum_{\frac{n\pi}{a},\, n=1,2,\ldots} A_n \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a} x}$$

condition 3 $\quad V(0,y) = V_0(y)$

$$\underbrace{\sum_{\frac{n\pi}{a},\, n=1,2,\ldots} A_n \sin\left(\frac{n\pi}{a} y\right)}_{\to \text{ complete set}}$$

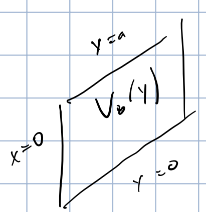
$$\int_0^a V_0(y) \cdot \sin\left(\frac{m\pi}{a} y\right) dy = \sum_{n=1} A_n \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dy = \frac{a}{2} \delta_{m,n}$$

$$= A_m \frac{a}{2}$$

$$A_m = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{m\pi}{a} y\right) dy$$

Sol^n $\quad V(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a} x}$

$$A_n = \frac{2}{a} \int_0^a V_0(y) \cdot \sin\left(\frac{n\pi}{a} y\right) dy$$



metal = conductor, inside $\vec{E} = 0 \to V$: const.

$$V_0(y) \equiv V_0$$

$$A_n = \frac{2}{a} V_0 \int_0^a \sin\left(\frac{n\pi}{a} y\right) dy$$

$$= -\frac{2}{n\pi} V_0 \left(\cos(n\pi) - 1\right) \quad \begin{cases} 0 & n = 2, 4, \ldots \\ 4V_0/n\pi & n = 1, 3, \ldots \end{cases}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n} \cdot \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a}x}$$

Other coords

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

most cases have azimuthal symmetry $\rightarrow \frac{\partial V}{\partial \phi} = 0$

$$\nabla^2 V = \frac{1}{r^2} \underline{\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)} + \frac{1}{r^2 \sin\theta} \underbrace{\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)}_{*}$$

$$V(r,\theta) = R(r) \cdot T(\theta)$$

$* \quad T(\theta) \cdot \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$

$* \quad R(r) \cdot \frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right)$

$$\nabla^2 V = 0 \rightarrow \quad 0 = \underbrace{\frac{1}{R} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{k} + \underbrace{\frac{1}{T} \frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right)}_{-k}$$

$$2r \frac{dR}{dr} + r^2 \frac{d^2R}{dr^2} = kR$$

Anzats: $R(r) = \alpha r^l$

$$\frac{d}{dr}(R) = \alpha l r^{l-1}$$
$$\frac{d^2}{dr^2}(R) = \alpha l (l-1) r^{l-2}$$

$\vdots$

$$\alpha(l(l-1) + 2l) = \alpha k$$

αr^l works if $k = l(l+1)$

another sol^n can be $R(r) = A_l r^l + B_l \frac{1}{r^{l+1}}$ & $k = l(l+1)$

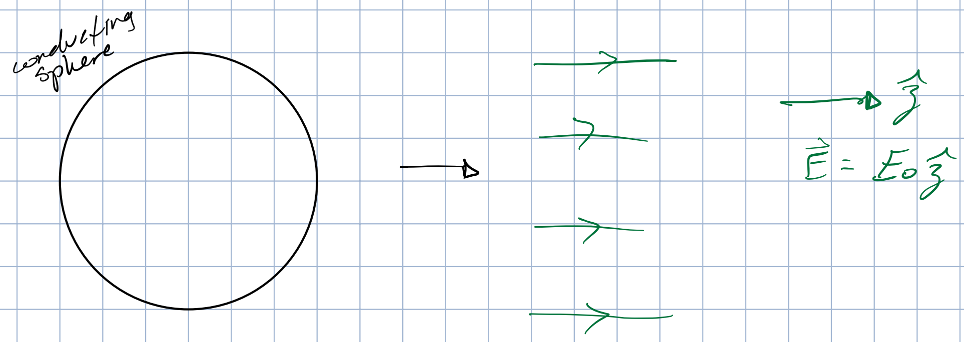
can do same w/ $T(\theta)$ $\rightarrow$ Legendre Polynomials

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

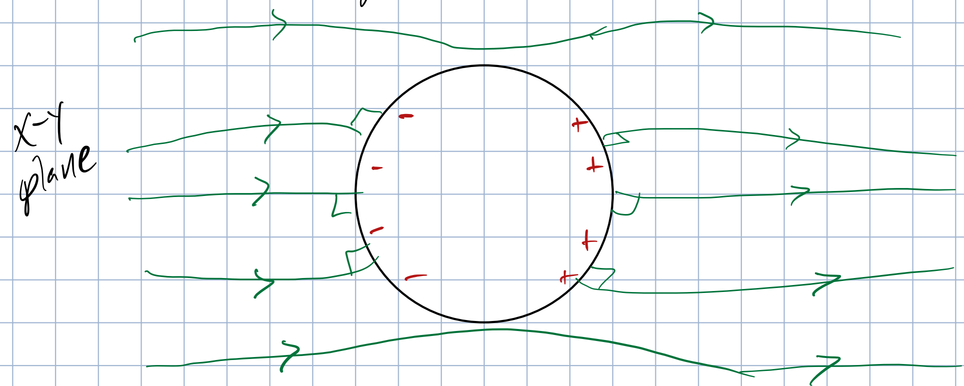
$$V(r,\theta) = \sum_l R_l(r) \cdot T_l(\cos\theta)$$

$$= \sum_{l=0}^{\infty} \left(A_l \cdot r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

Example: neutral metal conducting sphere in external uniform $\vec{E}$. find V outside



azimuthal symmetry



Inside conductor : induced charge: $\vec{E} = 0$ equipotential : $V_{conductor} = 0$

Outside conductor

near conductor, induced charge distorts $\vec{E}$ field

x-y plane ($z=0$) : $V=0$

far from conductor (large z): same as $\vec{E}_{ext}$ $\rightarrow V = -E_0 z + C$

$V_0 = -E_0 \cdot 0 + C = 0 \rightarrow C = 0$

$\rightarrow V = -E_0 z = -E_0 r\cos\theta$

general sol^{n}: $V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell \cdot r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

Boundary conditions : ① $V(r=a,\theta) = 0$ $V(r\to\infty,\theta) = -E_0 r\cos\theta$

① $V(a,\theta) = \sum_{\ell=0}^{\infty} \underbrace{\left(A_\ell \cdot a^\ell + \frac{B_\ell}{a^{\ell+1}}\right)} P_\ell(\cos\theta) = 0$

$\rightarrow B_\ell = -A_\ell \; a^{2\ell+1}$

$$V(r,\theta)=\sum_{\ell=0}^{\infty} A_\ell\left(r^\ell-\frac{a^{2\ell+1}}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

② $$V(\infty,\theta)=\sum_{\ell=0}^{\infty} A_\ell\left(\infty^\ell-\cancel{\frac{a^{2\ell+1}}{\infty^{\ell+1}}}^{\,0}\right)P_\ell(\cos\theta)$$

$$=\sum A_\ell r^\ell P_\ell(\cos\theta)$$

$$=\left(A_0+A_1 r\cos\theta + A_2 r^2\left[\tfrac{1}{2}(3\cos^2\theta-1)\right]+\dots\right)$$

$$A_0=0 \quad A_1=-E_0 \qquad A_{2,3,\dots}=0$$

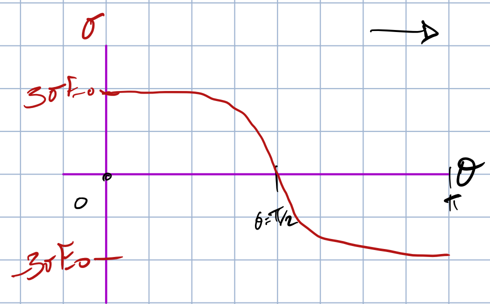
$$V(r,\theta)=-E_0\left(r-\frac{a^3}{r^2}\right)\cos\theta$$

induced charge σ on surface, E is $\perp$ to surface

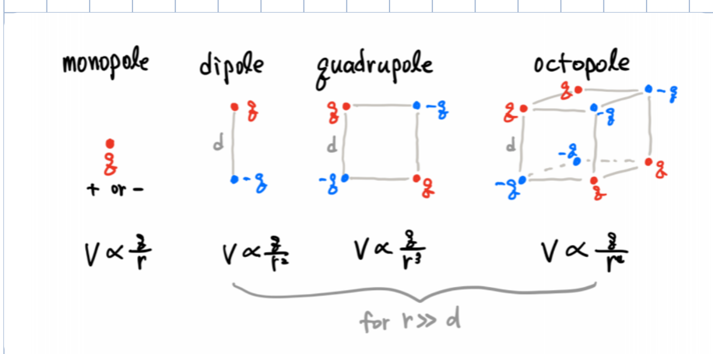
$$\vec{E}=E\,\hat{r} \qquad \vec{E}=\frac{\sigma}{\epsilon_0}\hat{r}=\left(-\frac{\partial V}{\partial r}\right)_{r=a}\hat{r}$$

$$-\frac{\partial V}{\partial r}=E_0\left(1+2\frac{a^3}{r^3}\right)\cos\theta\Big]_{r=a}=\frac{\sigma}{\epsilon_0}$$

$$\longrightarrow \sigma=3\epsilon_0\cdot E_0\cos\theta$$



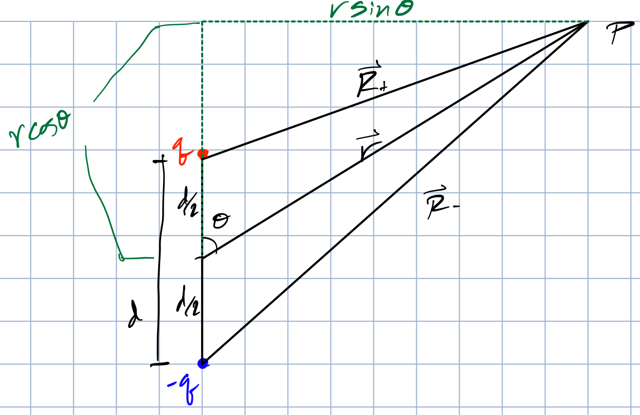
Multipoles!



Dipole

potential of a dipole: superposition of 2 point charges

exact sol^{n}: $\quad V(\vec{r})=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_+}-\frac{q}{R_-}\right)$



$$R_{\pm} = (r\sin\theta)^2 + (r\cos\theta \mp \tfrac{d}{2})^2$$