Laplace Eq$^{\underline{n}}$ & Image Charges

Laplace Eq$^{\underline{n}}$ (G. 3.1)

integrals harder than derivatives
still need boundary conditions

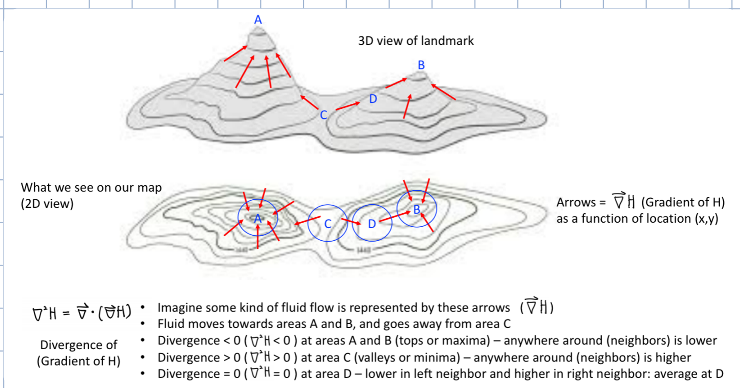
$\vec{\nabla} \cdot (\vec{\nabla} V)$ — divergence of (gradient of scalar)

gradient: points to peaks in value

divergence: how much does it act like sink/hole

div > 0 — div < 0 — div = 0

average



local min $\nabla^2 < 0$
local max $\nabla^2 > 0$

$f(x)$ is avg around $\Delta x + x$

Laplace picks out smooth area

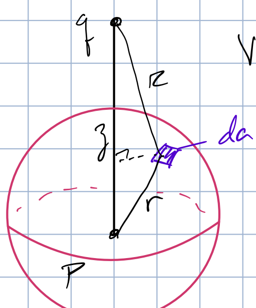
In 2D: if V is a sol$^{\underline{n}}$: $V(x,y) = \frac{1}{2\pi r} \oint V(\text{circle})\, dl$ (circle circumference)

in 3D: $V(x,y,z) = \frac{1}{4\pi r^2} \oint V(\text{surface}) \cdot da$ (sphere surface area)

↓

$V(x,y,z)$ is avg around (x,y,z)

example



$V(P) = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{z}$

$\rho(P) = 0 \rightarrow$ V@ P satisfies $\vec{\nabla}^2 V(\vec{r}) = 0$

V@ P is avg of V values around P

V@ P should be $V(P) = \frac{1}{4\pi r^2} \oint V(\text{surface})\, da$

V @ P: $V(d_a) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R}$

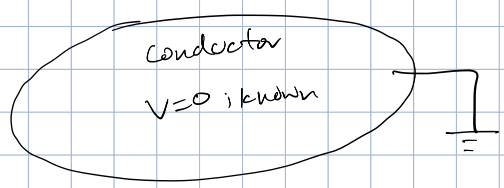
$$R = \sqrt{(r\sin\theta)^2 + (z - r\cos\theta)^2}$$

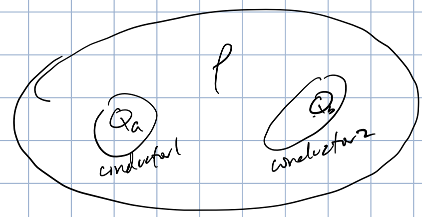
Boundary Conditions & Uniqueness Th^{m}

① Uniqueness on Electric Potential

if V is specified on boundary surface S, sol^{n} to Laplace in Volume V is uniquely determined

② Conductors

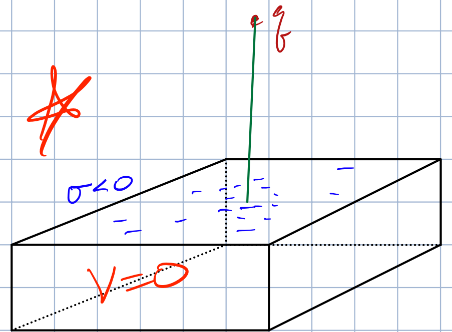




$\vec{E}$ is uniquely determined

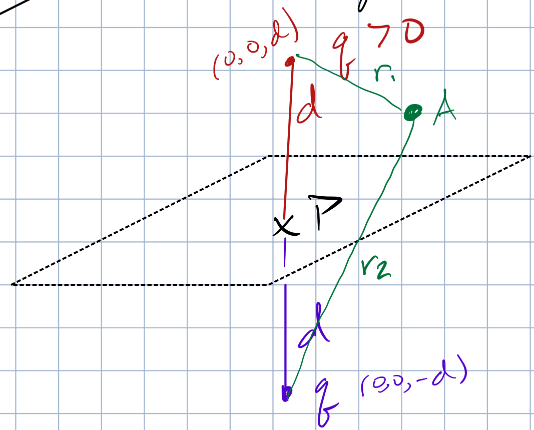
only 1 way for charges to be distributed

Method of Images



σ not uniform

induced charge



z > 0 real

z = 0 imaginary plane

z < 0 imaginary

$V = V_q + V_{-q}$

$r_1^2 = x^2 + y^2 + (d - z)^2$

$r_2^2 = x^2 + y^2 + (d + z)^2$

$z \to 0$, $z \to \infty$ ≥ 0

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right)$$

electric potential produced by setup ★

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z} = -\vec{\nabla} V \quad @ z=0^+ = -\frac{\partial V}{\partial z}$$

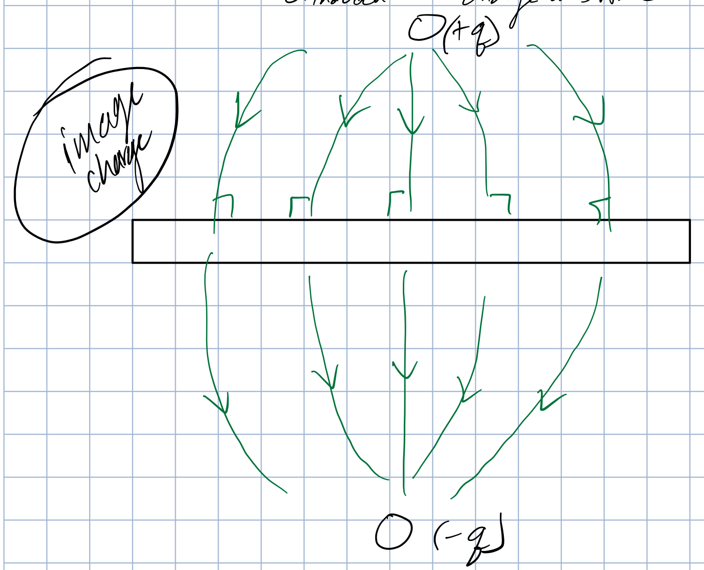
$$\sigma = \frac{-qd}{2\pi(x^2+y^2+d^2)}$$

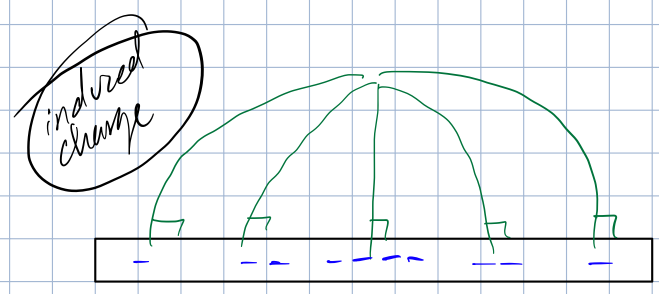
if $q > 0 \rightarrow \sigma < 0$

σ is max @ $x=y=0$

Total charge induced? Guess $= -q$

$Q_{induced}$ = charge on surface = $\int \sigma \, da$

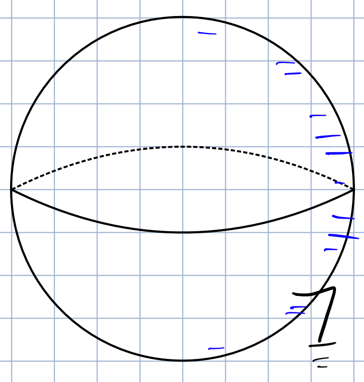




force? $\vec{F}_{induced} = q \cdot \vec{E}_{induced} \rightarrow -\vec{\nabla} V(\text{induced}) \rightarrow -\vec{\nabla} V(\text{image}) \rightarrow \vec{E}(\text{image})$

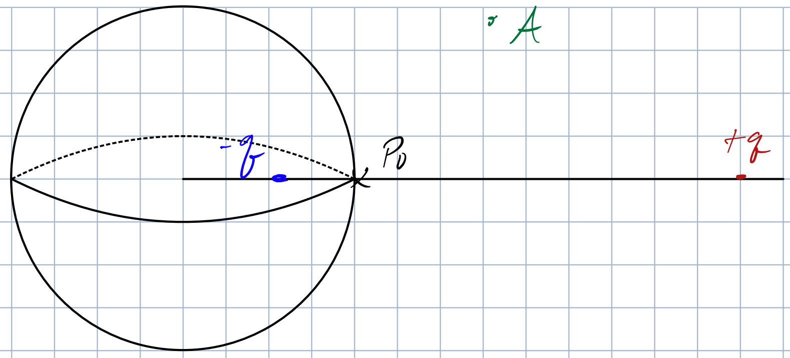
$$= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

• A



• $+q > 0$

A



① Draw line between q & center of conductor

② Put imaginary charge on line