

Laplace Eqⁿ & Image Charges

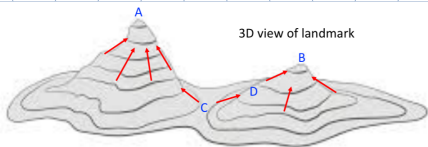
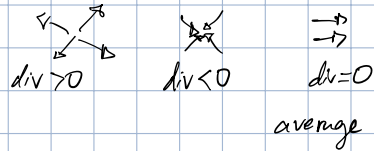
Laplace Eqⁿ (G. 3.1)

integrals harder than derivatives
still need boundary conditions

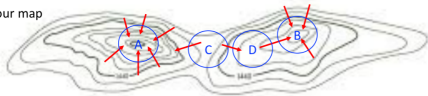
$\nabla \cdot (\nabla V)$ divergence of (gradient of scalar)

gradient: points to peaks in value

divergence: how much does it act like sink/hole



What we see on our map (2D view)



Arrows = $\vec{\nabla} H$ (Gradient of H) as a function of location (x,y)

- $\nabla \cdot H = \vec{\nabla} \cdot (\vec{\nabla} H)$
- Imagine some kind of fluid flow is represented by these arrows ($\vec{\nabla} H$)
 - Fluid moves towards areas A and B, and goes away from area C
 - Divergence < 0 ($\nabla \cdot H < 0$) at areas A and B (tops or maxima) - anywhere around (neighbors) is lower
 - Divergence > 0 ($\nabla \cdot H > 0$) at area C (valleys or minima) - anywhere around (neighbors) is higher
 - Divergence $= 0$ ($\nabla \cdot H = 0$) at area D - lower in left neighbor and higher in right neighbor: average at D

local min $\nabla^2 < 0$
local max $\nabla^2 > 0$

fix \vec{x} is avg around $\delta x + \delta x$

Laplace picks out smooth area

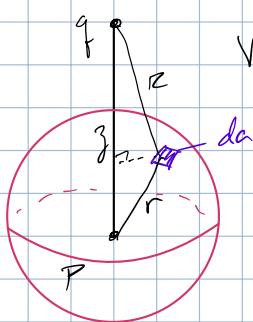
in 2D: if V is a solⁿ: $V(x,y) = \frac{1}{2\pi r} \oint V(\text{circle}(r)) dl$

in 3D: $V(x,y,z) = \frac{1}{4\pi r^2} \oint V(\text{surface}) da$

$V(x,y,z)$ is avg around (x,y,z)

circle circumference
sphere surface area

example



$$V(P) = \frac{q}{4\pi \epsilon_0} \cdot \frac{1}{r}$$

$\rho(P) = 0 \rightarrow \forall \epsilon$ P satisfies $\nabla^2 V(P) = 0$

$\forall \epsilon$ P is avg of V values around P

$\forall \epsilon$ P should be $V(P) = \frac{1}{4\pi r^2} \oint V(\text{surface}) da$

$$V \text{ e } P: V(da) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}$$

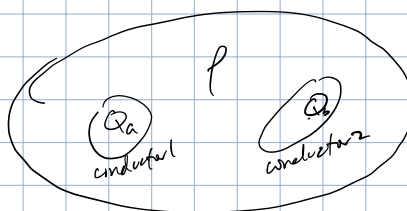
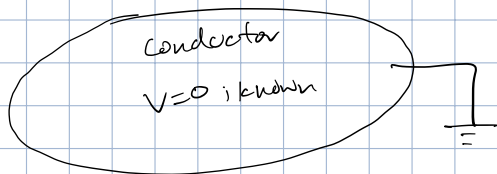
$$R = \sqrt{(r \sin \theta)^2 + (z - r \cos \theta)^2}$$

Boundary Conditions & Uniqueness Theorem

① Uniqueness on Electric Potential

if V is specified on boundary surface S , solⁿ to Laplace in Volume V is uniquely determined

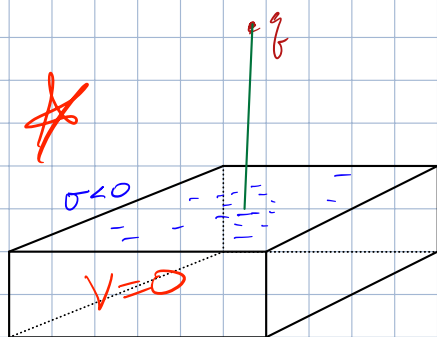
② Conductors



\vec{E} is uniquely determined

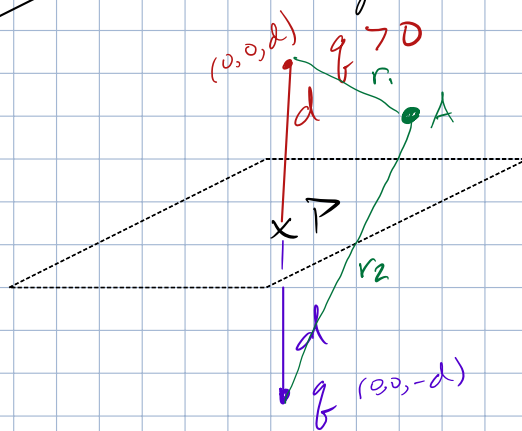
only 1 way for charges to be distributed

Method of Images



σ not uniform

induced charge



$z > 0$ real

$z = 0$ imaginary plane

$z < 0$ imaginary

$$V = V_q + V_{-q}$$

$$r_1^2 = x^2 + y^2 + (d-z)^2$$

$$r_2^2 = x^2 + y^2 + (d+z)^2$$

$$\begin{matrix} z \rightarrow 0 \\ z \rightarrow \infty \end{matrix} \geq 0$$

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_1} - \frac{q}{r_2} \right)$$

electric potential produced by setup *

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z} = -\nabla V \text{ @ } z=0 = -\frac{\partial V}{\partial z}$$

$$\sigma = \frac{-q d}{2\pi(x^2 + y^2 + d^2)}$$

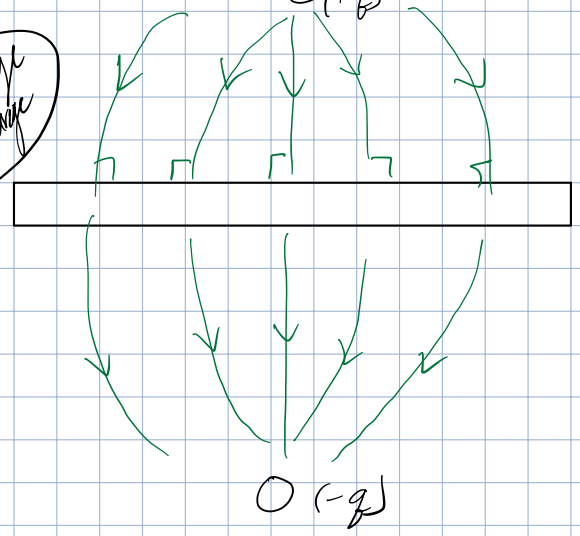
if $z > 0 \rightarrow \sigma < 0$

σ is max @ $x=y=0$

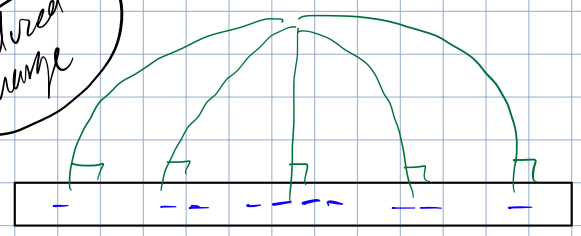
Total charge induced? Guess = $-q$

$$Q_{\text{induced}} = \text{charge on surface} = \int \sigma da$$

image charge

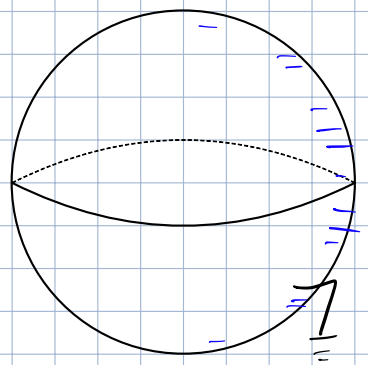


induced charge



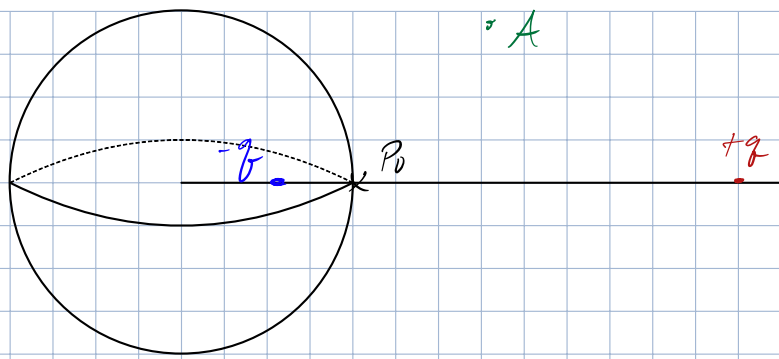
force? $\vec{F}_{\text{induced}} = q \cdot \vec{E}_{\text{induced}}$
 $= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{z}$
 • A

$-\nabla V(\text{induced}) \rightarrow -\nabla V(\text{image}) \rightarrow \vec{E}(\text{image})$



• $+q > 0$

))



① Draw line between q & center of conductor

② Put imaginary charge on line

