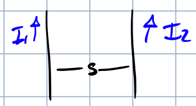
Electrostatics & Magnetostatics   Maxwell's eqns

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

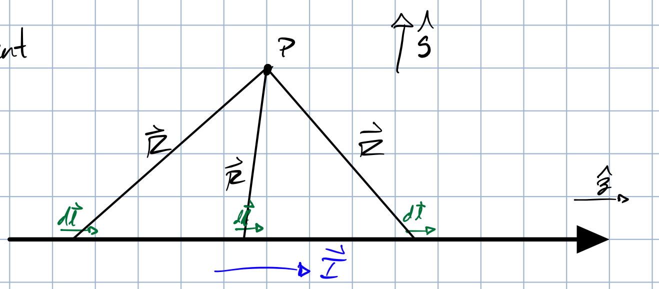
$$\xrightarrow[\frac{\partial\vec{B}}{\partial t} = 0]{\frac{\partial\vec{E}}{\partial t} = 0}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla\times\vec{E} = 0$$
$$\nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{B} = \mu_0\vec{J}$$

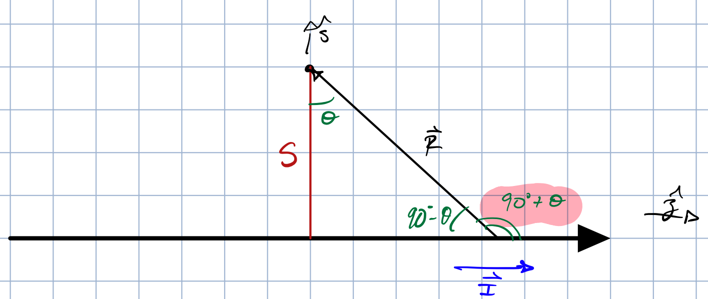
from observations: $F = \text{const}\cdot I_1\cdot I_2\,\frac{1}{s}$



Look @ current



we can write $\vec{B}$ as $\vec{B} = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{d\vec{l}\times\vec{R}}{R^2} \longrightarrow \vec{B} = \frac{\mu_0}{4\pi}\frac{I}{s}\hat{\phi}$



$$d\vec{l} = dz\,(\hat{z})$$

$$S = R\cos\theta \rightarrow R = \frac{S}{\cos\theta}$$

$$z = s\tan\theta \rightarrow \frac{dz}{d\theta} = s\cdot\frac{d}{d\theta}(\tan\theta) = s\cdot\sec^2\theta = \frac{s}{\cos^2\theta}$$
$$\rightarrow dz = \frac{s}{\cos^2\theta}\cdot d\theta$$

$$d\vec{l}\times\vec{R} = dl\sin(90°+\theta)\cdot\hat{\phi} \quad \text{(out of page)}$$
$$= dl\cos\theta\,\hat{\phi}$$
$$= dz\cos\theta\,\hat{\phi} = \frac{s}{\cos^2\theta}\,d\theta\cdot\cos\theta\,\hat{\phi} = \frac{s}{\cos\theta}\cdot d\theta\,\hat{\phi}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{d\vec{l}\times\vec{R}}{R^2} = \frac{\mu_0}{4\pi} I \int_{\theta=-90°}^{90°}\frac{\frac{s}{\cos\theta}\cdot d\theta}{\left(\frac{s}{\cos\theta}\right)^2}\,\hat{\phi}$$
$$= \frac{\mu_0}{4\pi}\frac{I}{s}\cdot\hat{\phi}\int_{\theta=-\pi/2}^{\pi/2}\cos\theta\,d\theta$$
$$= \frac{\mu_0}{4\pi}\frac{I}{s}\cdot\hat{\phi}\cdot\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{\mu_0}{2\pi}\frac{I}{s}\cdot\hat{\phi}$$

magnetic field: $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{R}}{R^2} \, dl$

Biot-Savart Law

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{R}}{R^2} \, d\tau' \quad \text{volume current}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \hat{R}}{R^2} \, da' \quad \text{surface current}$$

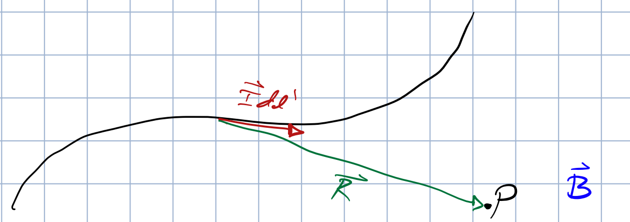
$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{R}}{R^2} \, dl' \quad \text{line current}$$

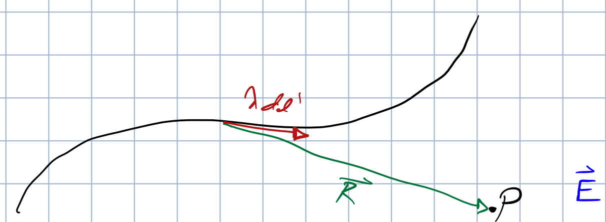
Colomb's Law

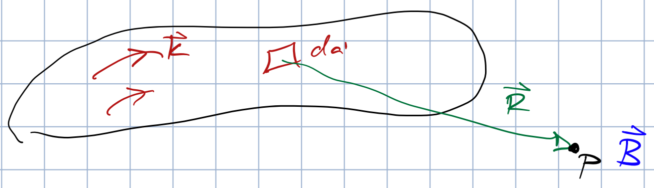
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho \hat{R}}{R^2} \, d\tau' \quad \text{volume charge}$$

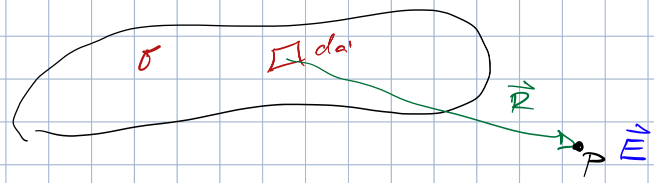
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma \hat{R}}{R^2} \, da' \quad \text{surface charge}$$

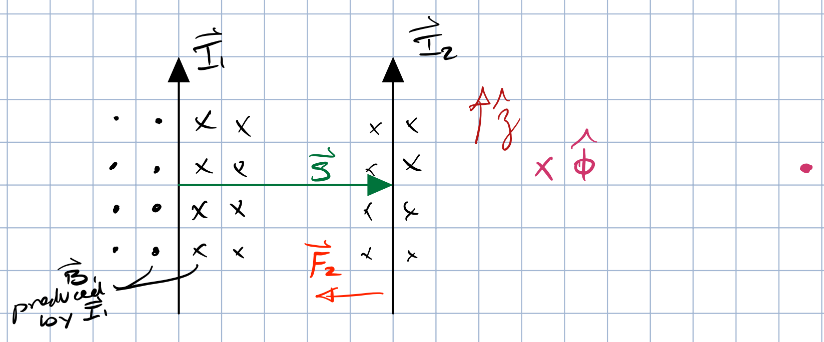
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda \hat{R}}{R^2} \, dl' \quad \text{line charge}$$

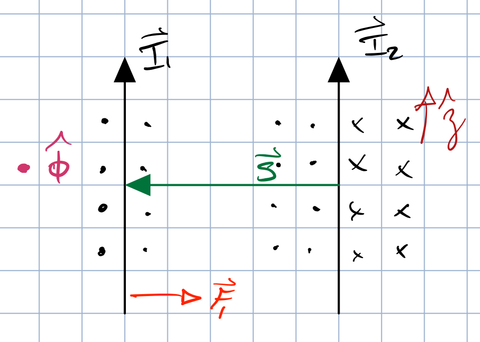












force on wire 2: $\vec{F}_2 = \int \vec{I}_2 \times \vec{B}_1 \, d\vec{l}_2$

$$= \int \vec{I}_2 \times \left(\frac{\mu_0 I_1}{2\pi s}\right) \hat{\phi} \, d\vec{l}_2$$

$$= \frac{\mu_0}{2\pi} \cdot \frac{I_1 I_2}{s} \int (-\hat{s}) \cdot d\vec{l}_2$$

force per unit length $\vec{f}_2 = -\frac{\mu_0 I_1 I_2}{2\pi s} \hat{s}$

force on wire 1: $\vec{F}_1 = \int (\vec{I}_1 \times \vec{B}_2) \, d\vec{l}_1$

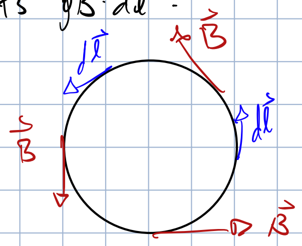
$$= \int \vec{I}_1 \times \left(\frac{\mu_0 I_2}{2\pi s}\right) \hat{\phi} \cdot d\vec{l}_1$$

$$= \frac{\mu_0}{2\pi} \cdot \frac{I_1 \cdot I_2}{s} \int (-\hat{s}) \, d\vec{l}_2$$

force per unit length $\vec{f}_1 = -\frac{\mu_0 I_1 \cdot I_2}{2\pi s} \hat{s}$

Divergence & Curl of $\vec{B}$

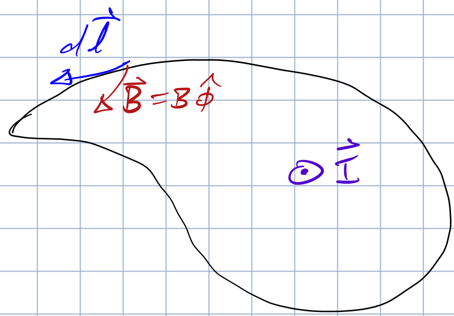
what's $\oint \vec{B} \cdot d\vec{l}$?



$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$d\vec{l} = s\,d\phi\,\hat{\phi}$$

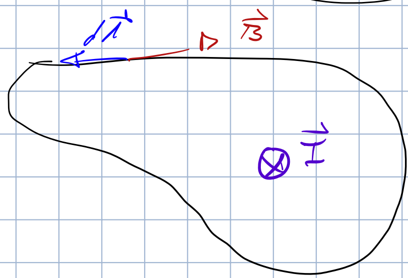
$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s}\hat{\phi} \cdot s\,d\phi\,\hat{\phi} = \frac{\mu_0 I}{2\pi} \cdot \oint d\phi = \mu_0 I$$



$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

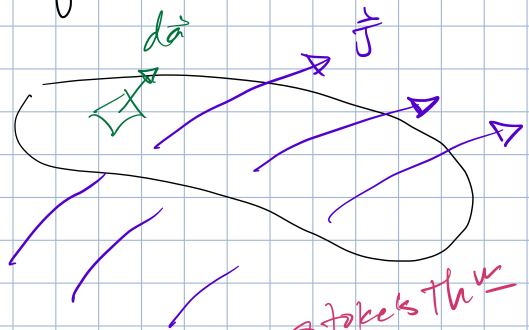
$$\vec{B} = B\,d\hat{\phi}$$

$$\oint \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \cdot \oint d\phi = \mu_0 I$$



$$\oint \vec{B} \cdot d\vec{l} = -\mu_0 I$$

in general, $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl.}$



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

stokes thm

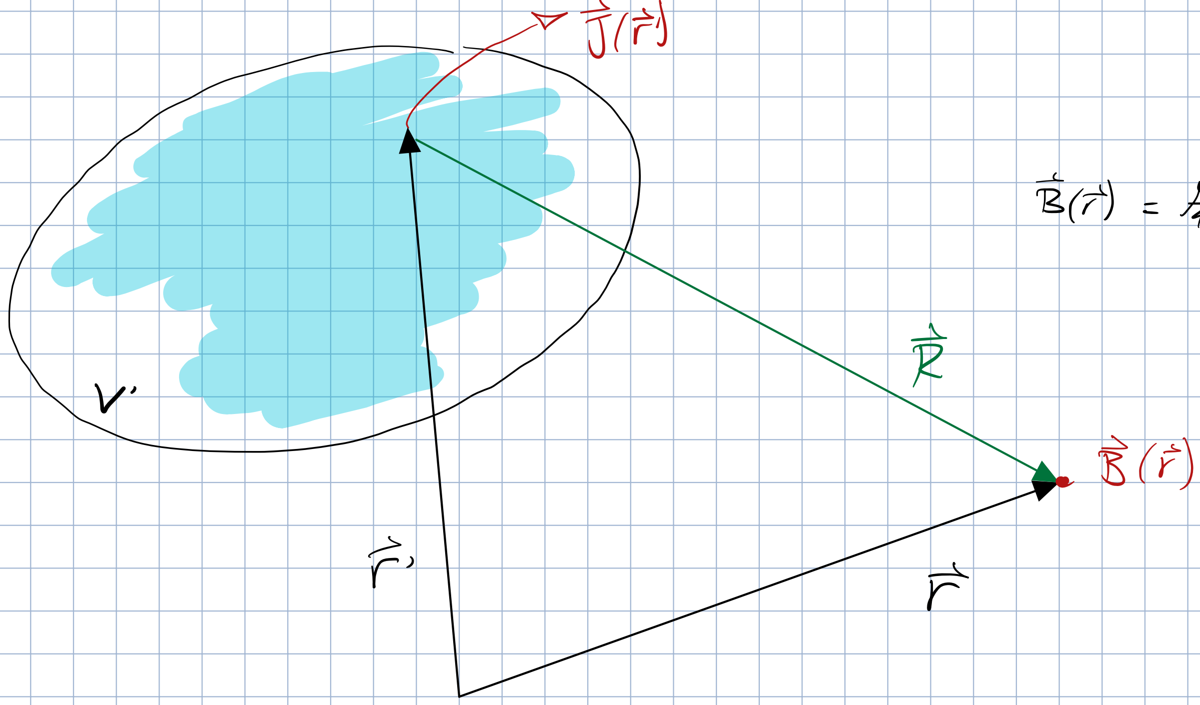
$$\oint_l \vec{B} \cdot d\vec{l} = \int_s (\vec{\nabla} \times \vec{B})\, d\vec{a} = \mu_0 \int_s \vec{J} \cdot d\vec{a} = \int_s (\mu_0 \vec{J})\, d\vec{a}$$

divergence thm

Ampere's Law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$$\oint_s \vec{E} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{E})\, d\tau = \frac{1}{\epsilon_0} \cdot \int_V \rho\, d\tau = \int_V \frac{\rho}{\epsilon_0}\, d\tau$$

Gauss's Law: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{R}}}{\mathscr{R}^2}\,d\tau'$$

i) $\vec{\nabla}\cdot\vec{B} = ?$

$$= \vec{\nabla}\cdot\left[\frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{R}}}{\mathscr{R}^2}\,d\tau'\right]$$

$$= \frac{\mu_0}{4\pi}\int_{V'} \vec{\nabla}\cdot\left[\underline{\vec{J}(\vec{r}')} \times \underline{\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}}\right]d\tau'$$

$\vec{\nabla}\times\vec{J}(\vec{r}') = 0$ $\quad$ $\vec{\nabla}(\vec{r})$ not $\vec{\nabla}(\vec{r}')$

$\vec{\nabla}\times\left(\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right) = 0$

$\longrightarrow \vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}\cdot(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

ii) $\vec{\nabla}\times\vec{B} = ?$

$\vec{\nabla}\cdot\vec{J}(\vec{r}') = 0$
(steady currents)

$$= \vec{\nabla}\times\left[\frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{R}}}{\mathscr{R}^2}\,d\tau'\right]$$

$$= \frac{\mu_0}{4\pi}\int_{V'} \vec{\nabla}\times\left(\vec{J}(\vec{r}') \times \frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right) d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{V'} d\tau'\left(\vec{J}(\vec{r}')\cdot\left(\vec{\nabla}\cdot\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right) - \left(\vec{J}(\vec{r}')\cdot\vec{\nabla}\right)\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right)$$

$$= \frac{\mu_0}{4\pi}\int_{V'} \vec{J}(\vec{r}')\cdot 4\pi\,\delta(\underbrace{\vec{\mathscr{R}}}_{\vec{r}-\vec{r}'})\,d\tau' - \underbrace{\frac{\mu_0}{4\pi}\int_{V'} \left(\vec{J}(\vec{r}')\cdot\vec{\nabla}\right)\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\,d\tau'}_{\to 0\ ?\ \text{slide 12}}$$

$$= \mu_0\,\vec{J}(\vec{r})$$

$$\vec{\nabla}\times\vec{B} = \mu_0\,\vec{J}(\vec{r})$$

Amperes Law: $\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})$ $\therefore$ $\oint \vec{B}\,d\vec{l} = \mu_0 \int \vec{J}(\vec{r})\,d\tau = \mu_0 I_{encl.}$

With stationary charges & steady currents

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ $\vec{E}$ field diverges away from positive charge

$\vec{\nabla}\times\vec{E} = 0$ $\vec{E}$ does not curl around

$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\cdot\int_{V'}\rho\,d\tau' = \frac{Q_{encl.}}{\epsilon_0}$ Gauss's Law

$\vec{\nabla}\cdot\vec{B} = 0$ $\vec{B}$ does not diverge

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ $\vec{B}$ curls around current

$\oint \vec{B}(\vec{r})\,d\vec{l} = \mu_0 \int_{V'} \vec{J}(\vec{r})\,d\tau' = \mu_0 I_{enc}$ Amperes Law