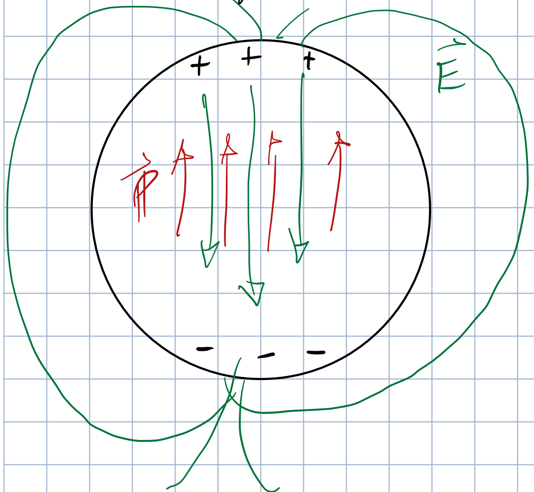
Have sphere w/ uniform charge

Potential & $\vec{E}$ by sphere radius a uniform polarization $=$ Potential & $\vec{E}$ by sphere radius a surface charge density $= P_0 \cos\theta$



$\vec{\mathcal{P}}$ = total dipole moment

$$\mathcal{P} = P_0 \tfrac{4}{3}\pi a^3$$

$$\frac{\mathcal{P}}{4\pi} = \frac{a^3 P_0}{3}$$

$$V_{in}(r,\theta) = \frac{P_0}{3\epsilon_0} r\cos\theta = \frac{P_0}{3\epsilon_0} z$$

$$\vec{E}_{in} = -\vec{\nabla}V_{in} = -\frac{\partial V}{\partial z} = -\frac{P_0}{3\epsilon_0}\hat{z}$$

$$V_{out}(r,\theta) = \frac{a^3}{3\epsilon_0}\frac{P_0\cos\theta}{r^2} = \frac{a^3}{3\epsilon_0}\frac{\vec{P}\cdot\hat{r}}{r^2} = \frac{\vec{\mathcal{P}}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = \frac{\mathcal{P}\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\vec{E}_{out} = \frac{\mathcal{P}}{4\pi\epsilon_0}\frac{1}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

Electric Displacement

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{1}{\mathscr{R}}\rho(\vec{r}\,')d\tau' \quad \text{free charges}$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{R}}}{\mathscr{R}^2} d\tau' \quad \text{pol.}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\int_{V'}\frac{\rho_{bound}(\vec{r}\,')}{\mathscr{R}}d\tau' + \int_{S'}\frac{\sigma_{bound}(\vec{r}\,')}{\mathscr{R}}da'\right)$$

$$\rho_{bound}(\vec{r}\,') = -\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')$$

$$\sigma_{bound}(\vec{r}\,') = \vec{P}(\vec{r}\,')\cdot\hat{n}'$$

$$\rho_{tot} = \rho = \rho_{free} + \rho_{bound} = \rho_{free} + (-\vec{\nabla}\cdot\vec{P})$$

$$\sigma_{tot} = \sigma = \sigma_{free} + \sigma_{bound} = \sigma_{free} + (\vec{P}\cdot\hat{n})$$

$$V = V_{free} + V_{bound}$$

$$\vec{E} = \vec{E}_{free} + \vec{E}_{bound}$$

Gauss's Law: $\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho = \frac{1}{\epsilon_0}(\rho_{free} - \vec{\nabla}\cdot\vec{P})$

$$\vec{\nabla}\vec{E} - \frac{1}{\epsilon_0}\vec{\nabla}\vec{P} = \frac{1}{\epsilon_0}\rho_{free}$$

$$\vec{\nabla}(\epsilon_0\vec{E} - \vec{P}) = \rho_{free}$$

$\vec{D} = \epsilon_0\vec{E} + \vec{P}$ electric displacement

$$\vec{\nabla}\cdot\vec{D} = \rho_{free}$$

Gauss Law integral

$$\int_V \vec{\nabla}\cdot\vec{D}\, d\tau = \int_V \rho_{free}\, d\tau$$

$$\oint_S \vec{D}\cdot d\vec{a} = Q_{free}^{encl.}$$

Curl of D

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\epsilon_0\vec{E}+\vec{P}) = \underbrace{\cancel{\epsilon_0\vec{\nabla}\times\vec{E}}}_{0} + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P}$$ may not be zero

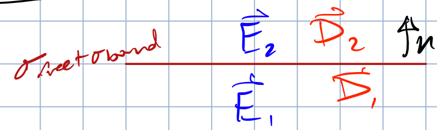
Boundary Conditions

$$\vec{\nabla}\cdot\vec{D} = \sigma_{free} \rightarrow D^{\perp}_{above} - D^{\perp}_{below} = \sigma_{free}$$

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P} \rightarrow D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\sigma_{free}+\sigma_{bound}}{\epsilon_0} \rightarrow E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma_{free}+\sigma_{bound}}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E} = 0 \rightarrow E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$



$$\sigma_{free} = (\vec{D}_2 - \vec{D}_1)\cdot\hat{n}$$

$$\sigma_{free} + \sigma_{bound} = \epsilon_0\,(\vec{E}_2 - \vec{E}_1)\cdot\hat{n}$$

Why use displacement $\vec{D}$?

can be simple when dealing w/dialectric since it can be deduced directly from free charge distribution

put dialectric into external field

→ polarization → modify total field → modify polarization → modify total field

in linear material: $\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E}$

$\vec{P} = \epsilon_0\chi_e\vec{E}$ ↳ electric susceptibility

$$= \epsilon_0\underbrace{(1+\chi_e)}_{\text{relative permittivity } \epsilon_r}\vec{E}$$

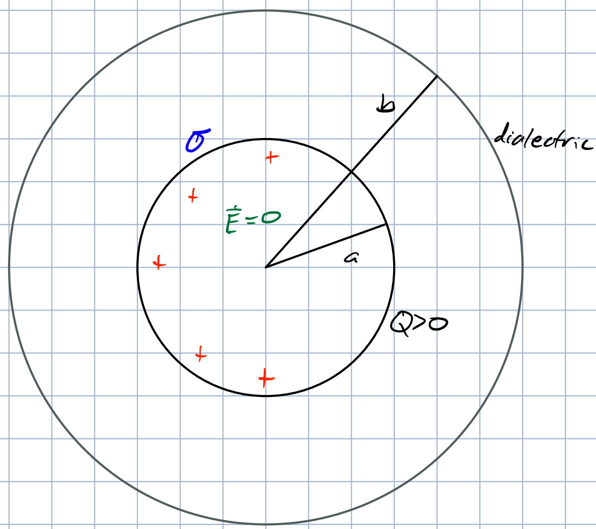
$$= \epsilon\,\vec{E}$$

↳ dialectric constant

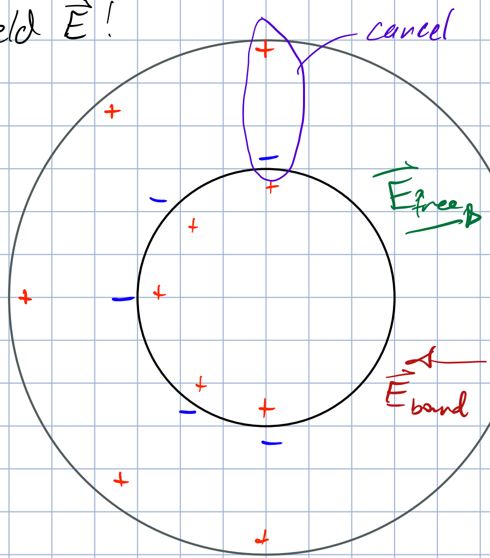
$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$ $\qquad \frac{\epsilon}{\epsilon_0}$ = no dims.

$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E}$ $\qquad \epsilon_r \geq 1$

Metal sphere surrounded by linear dielectric shell
$r=a$ $Q>0$ $\qquad r=b$ $Q=0$



external field $\vec{E}$!



$E_{free} = |\vec{E}_{free}| \cdot \hat{r}$

$\vec{E}_{bond} = |\vec{E}_{bond}| \cdot \hat{r}$

charge σ sperical symmetry

$\vec{E}_{free}$ spherical symmetry
$\longrightarrow$ polarized
$\longrightarrow$ $\vec{E}_{bond}$ spherical symmetry
$\longrightarrow$ $\vec{E} = \vec{E}_{free} + \vec{E}_{bond}$

$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E}$

$r<a \longrightarrow \vec{E}=0$ & $V=$ const.

$V(r=b) = \frac{Q}{4\pi\epsilon_0}\frac{1}{b}$

$\rho = 0 \qquad \sigma_{cond} = \frac{Q}{4\pi a^2}$

$a<r<b \longrightarrow \oint \vec{D}\cdot d\vec{a} = Q_{encl.}^{free}$

$\downarrow$

$4\pi r^2 \cdot D = Q$

$V(r=a) = -\int_{\infty}^{a} \vec{E}\, d\vec{l}$

$= -\int_{a}^{b} \vec{E}\cdot d\vec{l} - \int_{b}^{\infty} \vec{E}\cdot d\vec{l}$

$$D = \frac{Q}{4\pi r^2} \qquad\qquad = \frac{1}{\epsilon_r}\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} + (\epsilon_r - 1)\frac{1}{b}\right)$$

$$\sigma_{bound}^{in} = \frac{\epsilon_r - 1}{\epsilon_r}\frac{Q}{4\pi a^2}$$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} = \epsilon_0 \epsilon_r \vec{E}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\underbrace{\frac{1}{\epsilon_r}}_{\text{reduces } \vec{E}}\cdot\hat{r}$$

$$\vec{D} = \epsilon_0(\epsilon_r - 1)\vec{E} = \frac{Q}{4\pi r^2}\frac{\epsilon_r - 1}{\epsilon_r}\hat{r}$$

$r > b$

$$\oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E \qquad\qquad V(r<a) = V(r=a)$$

$$\frac{Q_{enc}}{\epsilon_0} = Q$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\cdot\hat{r} \qquad\qquad \sigma_{bound}^{out} = \frac{\epsilon_r - 1}{\epsilon_r}\frac{a^2}{b^2}\sigma_{bond}.$$