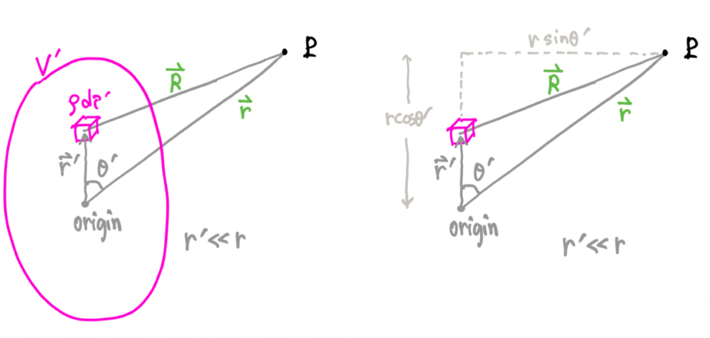
# Multipole

general case of continuous charge distribution




$\phi$ symmetry

$$R^2 = r^2 \sin^2\theta' + (r\cos\theta' - r')^2$$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)\right]$$

$$\frac{1}{R} = \frac{1}{r}\left[1 + \frac{r'}{r}\underbrace{\left(\frac{r'}{r} - 2\cos\theta\right)}_{x}\right]^{-1/2} \qquad \underline{\ll 1}$$

$$= \frac{1}{r}(1+x)^{-1/2}$$

$$\vdots$$

$$= \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\theta')$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{1}{R}\rho(\vec{r}')\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{V'} r'^n P_n(\cos\theta')\rho(\vec{r}')\,d\tau'$$

monopole: $\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_{V'}\rho(\vec{r}')\,d\tau$

dipole: $\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{V'} r'\cos\theta'\,\rho(\vec{r}')\,d\tau'$

quadrupole: $\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_{V'} r'^2 P_2(\cos\theta')\,\rho(\vec{r}')\,d\tau$

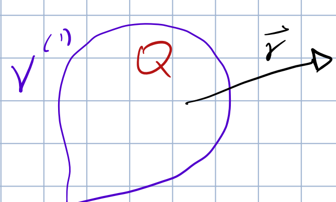
depends on origin

## Monopole

$$V^{(1)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_V \rho(\vec{r}')\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

total charge located @ origin



## Dipole

$$V^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{V'} \vec{r}' \cdot \hat{r}\, \rho(\vec{r}')\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \int_{V'} \vec{r}'\, \rho(\vec{r}')\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \vec{p} \cdot \hat{r}$$

$\vec{p} \doteq$ dipole moment

depends on charge distribution & origin

## Potential far from charges

$$\frac{1}{r} \gg \frac{1}{r^2} \gg \frac{1}{r^3} \ldots$$

if net charge $\neq 0$

$\longrightarrow$ monopole is dominant $\propto \frac{1}{r}$

if net charge $= 0$

$\longrightarrow$ dipole is dominant $\propto \frac{1}{r^2}$

$$V^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \vec{p} \cdot \hat{r} \frac{1}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} p\cos\theta$$

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$E_r = \frac{1}{4\pi\epsilon_0} \frac{2p\cos\theta}{r^2}$$

$$E_\theta = \frac{1}{4\pi\epsilon_0} \frac{p\sin\theta}{r^2}$$

$$E_\phi = 0$$

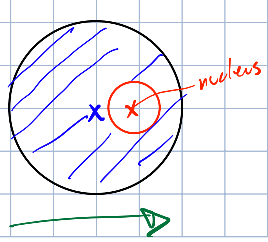
$$\vec{E}^{(2)} = \vec{E}^{(\text{dipole})} = \frac{1}{4\pi\epsilon_0} \frac{p}{r^2} (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta})$$

## Neutral Atom

no dipole moment

## Atom in external field

center of $e^-$ cloud $\neq$ center




$$\vec{p} = q\vec{d} \propto \vec{E}_{ext}$$

polarization = dipole / volume