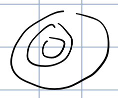
GR review



isotropic
not homogeneous

homogeneous
not isotropic

isotropy for all observers
implies homogeneity

Multi component Universes

for arbitrary partitions total energy density

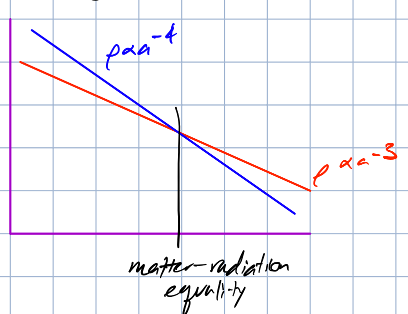
$$\rho_{TOT}(a) = \underbrace{\rho_{R,0}\, a^{-4}}_{\text{radiation}} + \underbrace{\rho_{m,0}\, a^{-3}}_{\text{dark matter}} + \underbrace{\rho_\Lambda}_{\text{constant}} , \qquad a(t_0) \equiv 1$$

$$= \rho_{TOT,0} \left[\Omega_R a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda \right]$$

fractional abundance: $\Omega_i \equiv \rho_{i,0} / \rho_{TOT,0}$

"critical density": $\rho_{crit} \equiv \rho_{TOT,0} = \frac{3H_0^2}{8\pi G} \approx 4 \cdot 10^{-47}\, \text{GeV}^4$

using $a = (1+z)^{-1} \rightarrow H(z) = H_0 \left[\Omega_R (1+z)^4 + \Omega_m (1+z)^3 + \Omega_\Lambda \right]^{1/2}$



Thermodynamics

Classical Phase Space — 6D phase space $(\vec{x}, \vec{p})$

phase space distribution f : $N = \int d^3\vec{x} \int d^3\vec{p}\; f(\vec{x}, \vec{p}, t)$

\# of particles in $dx\,dp$ region

f depends on forces & typical energies

QM in box

$V = L^3 \qquad \lambda = \frac{h}{q} = \frac{2\pi\hbar}{q}$ (de Broglie) periodicity: $e^{2\pi i x/\lambda} = e^{iqx} = e^{iq(x+L)} \rightarrow e^{iqL} = 1$

requires discrete spectrum of momenta: $q_i = \frac{2\pi}{L} n_i$

sum over all $\sum_{n_x n_y n_z} \rightarrow \int d^3n \qquad d^3n = \left(\frac{L}{2\pi}\right)^3 d^3q$

density of states (g spin) $\frac{dN}{V} = \frac{g}{V} d^3n = \frac{g}{(2\pi)^3} dq^3$

if all modes are filled up, $n = \frac{N}{V} = \frac{g}{(2\pi)^3} \int dq^3$

if phase-space distribution f of occupation numbers

$$n = \frac{N}{V} = \frac{g}{(2\pi)^3} \cdot \int dq^3 f(q,t)$$

Goal: apply equilibrium form of $f(q,t)$

Equilibrium

in early time, particle species were in common state of TE (Thermal Equilibrium) satisfying

① Kinetic Equlibrium — common temp. among all particles

② Chemical Equilibrium — balanced creation/annihilation rates

$\longleftrightarrow$

Phase Space Distribution

differential # density: $dn = f(\vec{p},t) \cdot \frac{d^3\vec{p}}{(2\pi)^3}$ (indp. of equilibrium)

in equilibrium: $f(|\vec{p}|,t) = \frac{g}{\exp((E-\mu)/T) \pm 1}$ $\quad E(p) = \sqrt{p^2+m^2}$

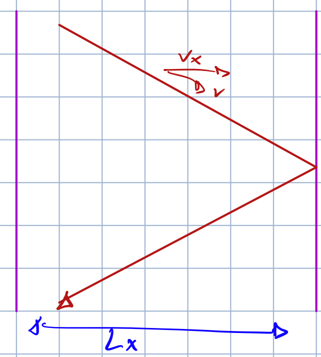
spin states

chemical potential

fermion/bosons

$$n = \int \frac{d^3p}{(2\pi)^3} f(\vec{p},t) \qquad \rho = \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot f(p,t) \cdot E(p)$$

Pressure



$$P = \frac{F}{A} = \frac{N}{A}\frac{\Delta p}{\Delta t}$$

$$\Delta p = 2|p_x| \qquad \Delta t = 2\frac{L_x}{v_x}$$

$$P = \frac{N}{A}\cdot\frac{2|p_x|}{2\frac{L_x}{v_x}} = \frac{N}{V}|p_x||v_x| = \frac{N}{V}\frac{|p||v|}{3}$$

using SR $v = (\gamma m v)/(\gamma m) = p/E$

$$P = \frac{N}{V}\frac{p^2}{3E}$$

averaging $P = \int \frac{d^3\vec{p}}{(2\pi)^3}\cdot\frac{p^2}{3E} f(p,t)$

relativistic $E >> m$

$$P \to \frac{1}{3}\int\frac{d^3\vec{p}}{(2\pi)^3} E\, f(p,t) \equiv \frac{1}{3}\rho$$

non relativistic $E << m$

$$P \to 0$$

equation of state parameter

$$w \equiv \frac{P}{\rho} = \frac{\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{3E}f(p,t)}{\int\frac{d^3p}{(2\pi)^3}E\cdot f(p,t)}$$

for a single species:

$$n(T) = \int\frac{d^3p}{(2\pi)^3}\cdot\frac{g}{\exp(E/T)\pm 1} = \frac{g}{2\pi^2}\int_0^\infty\frac{dp\,p^2}{e^{E/T}\pm 1}$$

no closed sol^{n} in general. take limit $E \sim T >> m$

zeta fx

$$n(T) = \frac{g}{2\pi^2}\int_0^\infty\frac{dp\,p^2}{e^{p/T}\pm 1} = a\cdot\frac{g\cdot\zeta(3)}{\pi^2}$$

$a=1$ (boses), $\frac{3}{4}$ (fermions)

non relativistic $\qquad T \ll m$

$$n(T) = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

as temp drops, heavy particles "Boltzmann" suppressed

Similar for energy density

$$\rho(T) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{g \cdot E}{\exp(E/T) \pm 1}$$

$$\rho_{TOT} = \sum_{\text{bosons}} \rho_{Boson} + \sum_{\text{fermion}} \rho_{Fermion}$$

(can have different temps.)

$$= \sum_i \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{g_i}{\exp(E/T_i) - 1} + \sum_j \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{g_j}{\exp(E/T_j) + 1} \equiv \frac{\pi^2}{30} \cdot g_*(T_\gamma) T_\gamma^4$$

$T \equiv T_\gamma$ (photon temp)

effective # of relativistic species

$$g_*(T_\gamma) \equiv \sum_i g_i \frac{\int d^3p \; E\left(e^{E/T_i} - 1\right)^{-1}}{\int d^3p \; \rho\left(e^{E/T_i} - 1\right)^{-1}} + \sum_j g_j \frac{\int d^3p \; E\left(e^{E/T_j} - 1\right)^{-1}}{\int d^3p \; \rho\left(e^{E/T_\gamma} - 1\right)^{-1}}$$

sum over $T > m$ (degrees of freedom)

$$g_*(T_\gamma) \approx \sum_{i=\text{bose}} g_i \left(\frac{T_i}{T_\gamma}\right)^4 + \frac{7}{8} \sum_{j=\text{fermion}} g_j \left(\frac{T_j}{T_\gamma}\right)^4$$

Fractional degree of freedom

Thermodynamics & Cosmic Expansion

FRW Temp. Evolution

temp is bulk measure of avg energy $T \propto \langle E \rangle$

for universe dominated w/ relativistic $T \gg m$, $E \approx p \rightarrow T \propto a^{-1}$

$$\frac{d\rho_r}{dE} = \frac{E^3}{\pi^2} \cdot \frac{1}{e^{E/T}-1} \longrightarrow \frac{E^3}{\pi^2} \frac{1}{e^{Ea/t}-1}$$

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 = \frac{2\zeta(3)}{\pi^2} T_0^3 a^{-3} \qquad \rho_\gamma = \frac{\pi^2}{15} T^4 = \frac{\pi^2}{15} T_0^4 a^{-4}$$

radiation dominated

$$\rho(T) = \frac{\pi^2}{30} g_*(T) \cdot T^4$$

$$H(T) = \left(\frac{8\pi G}{3}\rho\right)^{1/2} \propto \sqrt{g_*}\, T^2 \qquad \text{(Friedmann eq}^{\underline{n}}\text{)}$$

constant g_* $\rightarrow$ # species unchanging & $a(t) \propto t^{1/2}$ $\quad T(a) \propto a^{-1}$

when particle interaction rate is faster than Hubble expansion rate ($\Gamma \gg H$)
local TE implies entropy per co-moving volume is conserved

$$\underset{\text{entropy}}{S} = \underset{\substack{\text{entropy}\\\text{density}}}{s(T)}\, a^3 = \text{const}$$

2nd law of thermo. (in physical volume $V = a^3$)

$$T\,dS = dU + \overset{\text{pressure}}{P}\,dV = d(\rho\, dV) + P\,dV$$

in terms of entropy density: $$d[s(T) \cdot V] = \frac{d[\rho(T) \cdot V] + P(T)\,dV}{T}$$

$$s(T) = \frac{\rho(T) + P(T)}{T} \qquad T\frac{dP(T)}{dT} = \rho(T) + P(T)$$

Have equilibrium forms of $\rho(T)$ & $P(T)$

$$s(T) = \frac{\rho(T) + P(T)}{T} = \frac{2\pi^2}{45} g_{*,s}(T)\, T^3$$

effective entropic degrees of freedom

$$g_{*,s}(T) \equiv \sum_{\text{bose}} g_i \cdot \left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_j \cdot \left(\frac{T_j}{T_\gamma}\right)^3$$

if all species have common temp. $g_* = g_{*,s}$

Entropy Bookkeeping

@ high temp. $T \gg m_\phi$

no kinetic barrier for $\psi\psi \leftrightarrow \gamma\gamma$

@ low temp. $T \ll m_\phi$

$\psi\psi \to \gamma\gamma$ efficient

$\gamma\gamma \to \psi\psi$ few photons do this (Boltzmann suppression)

heavy species go extinct, but entropy survives & heats other particles

instantaneous depletion: $s_i a_i^3 = s_f a_f^3$

$$g^i_{*,s} \cdot T_i^3 = g^f_{*,s} \cdot T_f^3 \longrightarrow T_f = \left(\frac{g^i_{*,s}}{g^f_{*,s}} \right)^{1/3} T_i$$